



NE 112 *Linear algebra for nanotechnology engineering*

1.6.12 The roots of unity

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Outline

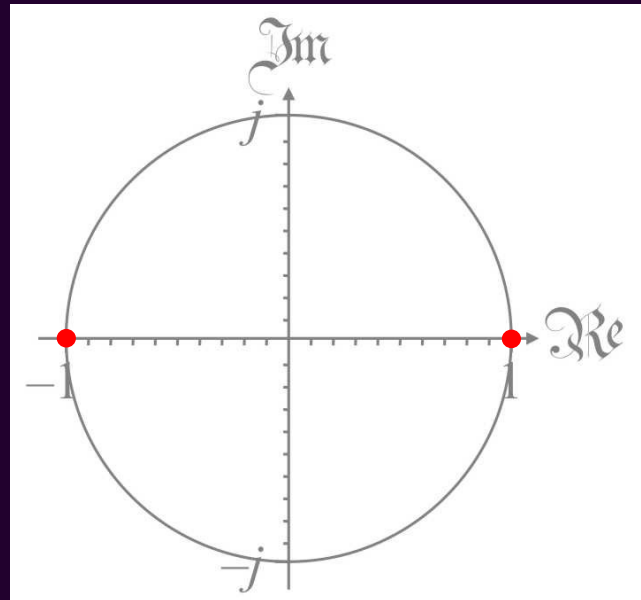
- In this topic, we will
 - Define the roots of unity
 - See how to find them





The 2nd roots of unity

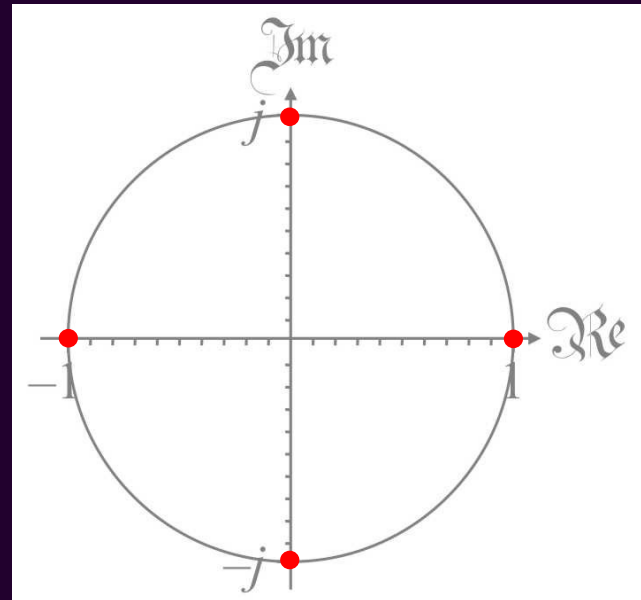
- The second roots of unity are all numbers which squared equal the value one
 - This includes $z = 1$ and $z = -1$
 - These are the roots of the polynomial $z^2 - 1$
 - There are exactly two roots





The 4th roots of unity

- The fourth roots of unity are all numbers which raised to the power four equal the value one
 - This includes $z = 1, j, -1$ and $-j$
 - These are the roots of the polynomial $z^4 - 1$
 - There are exactly four roots





The 3rd roots of unity

- The third roots of unity are all numbers which cubed equal the value one
 - This includes $z = 1$ but what else?
 - These are the roots of the polynomial $z^3 - 1$
 - There will be exactly three roots

- In the polar representation, note that

$$(1 \angle \theta)^3 = 1 \angle (3\theta)$$

- Thus, $3\theta = k(2\pi)$ or $\theta = k \frac{2\pi}{3}$





The 3rd roots of unity

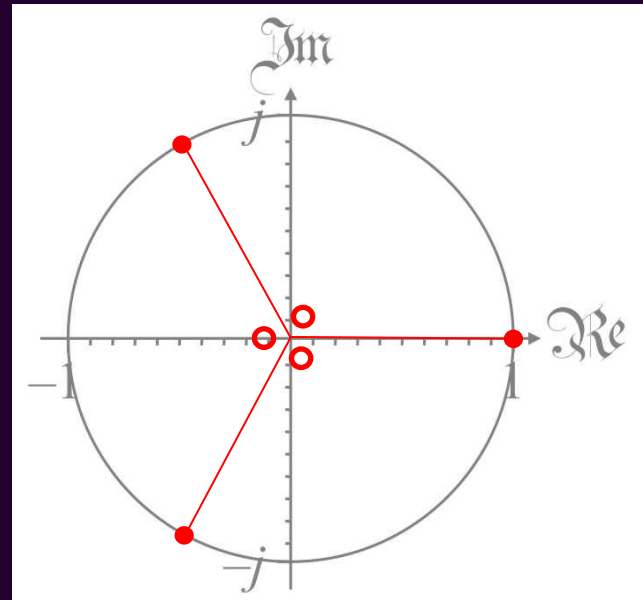
- Let us plot these when $\theta = k \frac{2\pi}{3}$

$$1 \angle \left(0 \frac{2\pi}{3} \right) = 1 \angle 0$$

$$1 \angle \left(1 \frac{2\pi}{3} \right) = 1 \angle \frac{2\pi}{3}$$

$$1 \angle \left(2 \frac{2\pi}{3} \right) = 1 \angle \frac{4\pi}{3}$$

$$1 \angle \left(3 \frac{2\pi}{3} \right) = 1 \angle (2\pi) = 1 \angle 0$$





The 5th roots of unity

- The fifth roots of unity are all numbers which raised to the power five equal the value one
 - These are the roots of the polynomial $z^5 - 1$
 - There will be exactly five roots

- In the polar representation, note that

$$(1 \angle \theta)^5 = 1 \angle (5\theta)$$

- Thus, $5\theta = k(2\pi)$ or $\theta = k \frac{2\pi}{5}$





The 5th roots of unity

- Let us plot these when $\theta = k \frac{2\pi}{5}$

$$1 \angle \left(0 \frac{2\pi}{5} \right) = 1 \angle 0$$

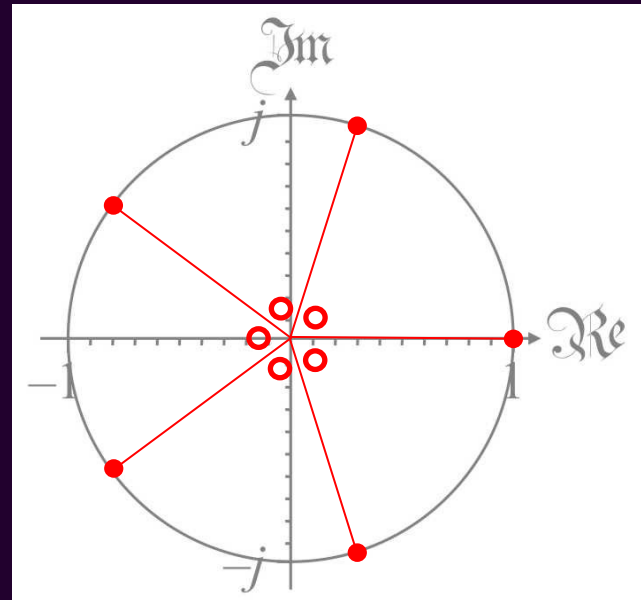
$$1 \angle \left(1 \frac{2\pi}{5} \right) = 1 \angle \frac{2\pi}{5}$$

$$1 \angle \left(2 \frac{2\pi}{5} \right) = 1 \angle \frac{4\pi}{5}$$

$$1 \angle \left(3 \frac{2\pi}{5} \right) = 1 \angle \frac{6\pi}{5}$$

$$1 \angle \left(4 \frac{2\pi}{5} \right) = 1 \angle \frac{8\pi}{5}$$

$$1 \angle \left(5 \frac{2\pi}{5} \right) = 1 \angle (2\pi) = 1 \angle 0$$





The n^{th} roots of unity

- The n^{th} roots of unity are all numbers which raised to the power n equal the value one
 - These are the roots of the polynomial $z^n - 1$
 - There will be exactly n roots

- In the polar representation, note that

$$(1\angle\theta)^n = 1\angle(n\theta)$$

- Thus, $n\theta = k(2\pi)$ or $\theta = k\frac{2\pi}{n}$





The n^{th} roots of unity

- Thus, the roots will have the arguments $\theta = k \frac{2\pi}{n}$

$$1 \angle \left(0 \frac{2\pi}{n} \right) = 1 \angle 0$$

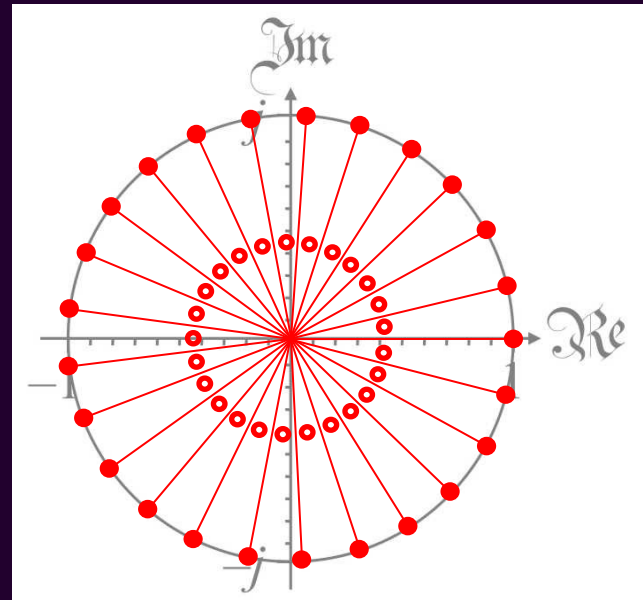
$$1 \angle \left(1 \frac{2\pi}{n} \right) = 1 \angle \frac{2\pi}{n}$$

$$1 \angle \left(2 \frac{2\pi}{n} \right) = 1 \angle \frac{4\pi}{n}$$

$$\vdots$$

$$1 \angle \left((n-1) \frac{2\pi}{n} \right) = 1 \angle \frac{2(n-1)\pi}{n}$$

$$1 \angle \left(n \frac{2\pi}{n} \right) = 1 \angle (2\pi) = 1 \angle 0$$





Application

- The n^{th} roots of unity are used in the discrete Fourier transform
 - This involves going from time or space domains to frequencies





Summary

- In this topic, you now
 - Understand the concept of the roots of unity
 - Have applied numerous ideas we have already discussed



References

- [1] https://en.wikipedia.org/wiki/Root_of_unity



Acknowledgments

None so far.



Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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